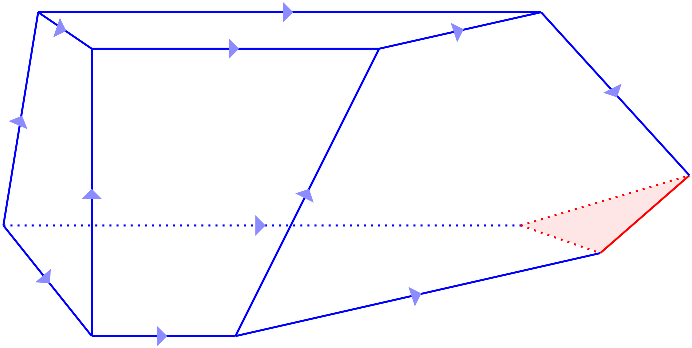
Number of paths per length on polytopes (counter)examples & central limit theorem

Germain Poullot & Martina Juhnke

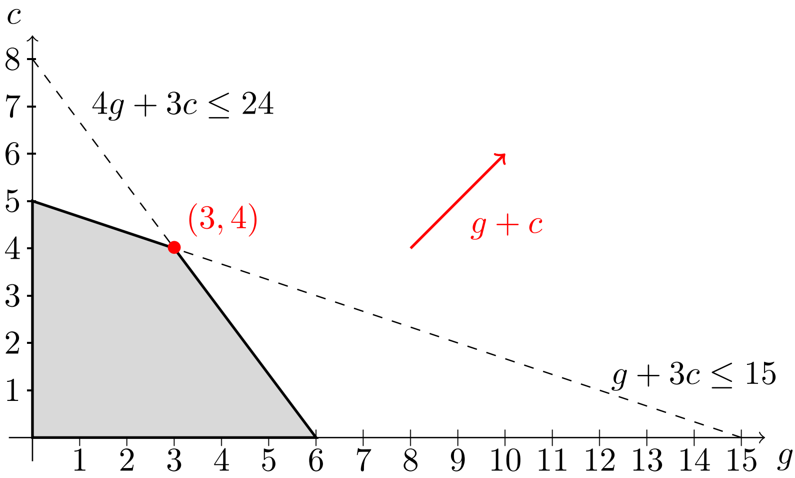


26 June 2025 arXiv:2504.20739

1. Monotone paths & coherent paths
   - Monotone paths
   - Coherent paths
   - Unimodality?
2. Positive examples
3. Negative examples
   - Lopsided d-cube
   - Simplicial
   - Generalized permutahedron
   - 0/1-coordinates
4. Random case
   - Uniform distribution on the sphere and β-polytopes
   - Expectancy
   - Variance
   - Central limit theorem

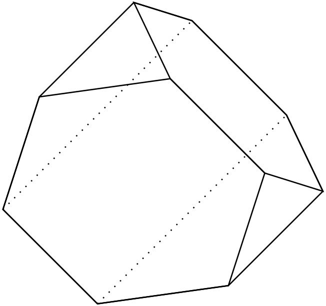
Monotone paths & coherent paths

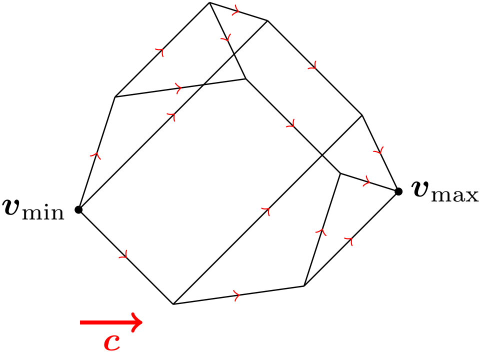
Linear constrains, linear objective function



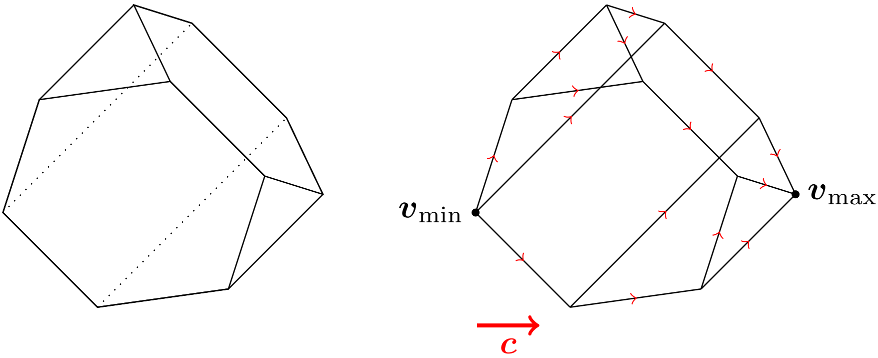
Solution: simplex method on polytope

Monotone paths





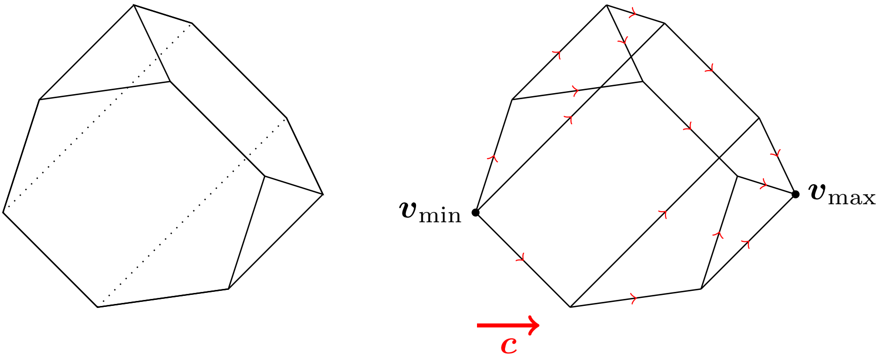
Monotone paths



ℓ	3	4	5	total
N_ℓ	3	2	2	7

Monotone path: directed path $\boldsymbol{v}_{\min} \rightsquigarrow \boldsymbol{v}_{\max}$ in directed graph $G_{\mathrm{P},\boldsymbol{c}}$

Monotone paths

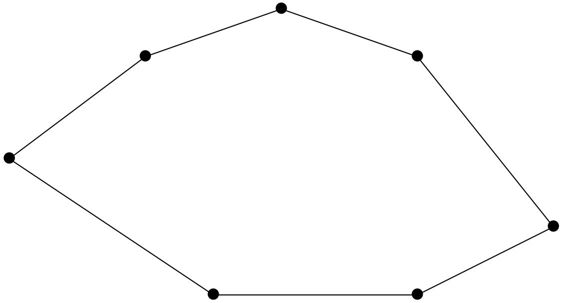


ℓ	3	4	5	total
N_ℓ	3	2	2	7

Monotone path: directed path $\boldsymbol{v}_{\min} \rightsquigarrow \boldsymbol{v}_{\max}$ in directed graph $G_{P,\boldsymbol{c}}$
length: number of edges
$N_\ell = \#\{\text{paths of length } \ell\}$

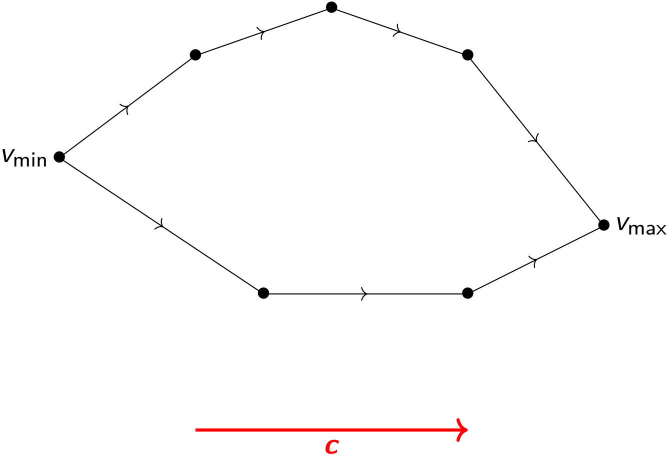
Shadow vertex (pivot) rule

Linear optimization in dimension 2 (simplex method):



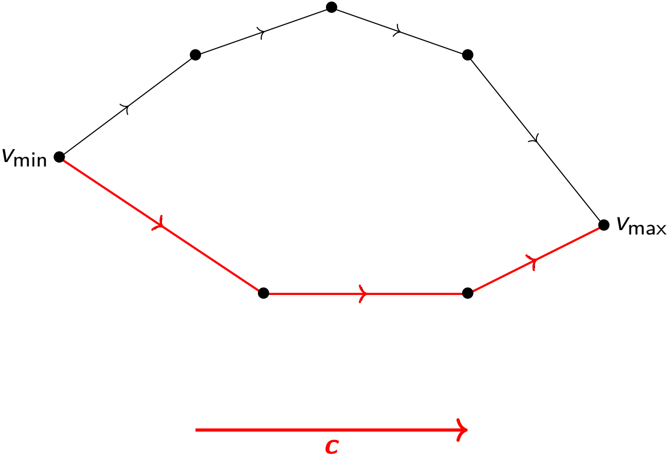
Shadow vertex (pivot) rule

Linear optimization in dimension 2 (simplex method):



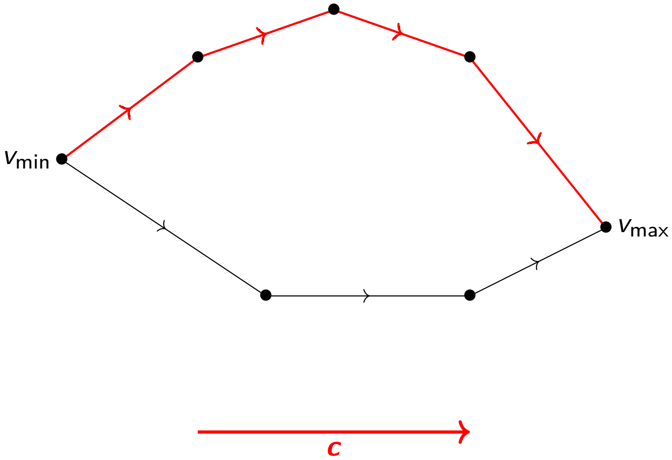
Shadow vertex (pivot) rule

Linear optimization in dimension 2 (simplex method):



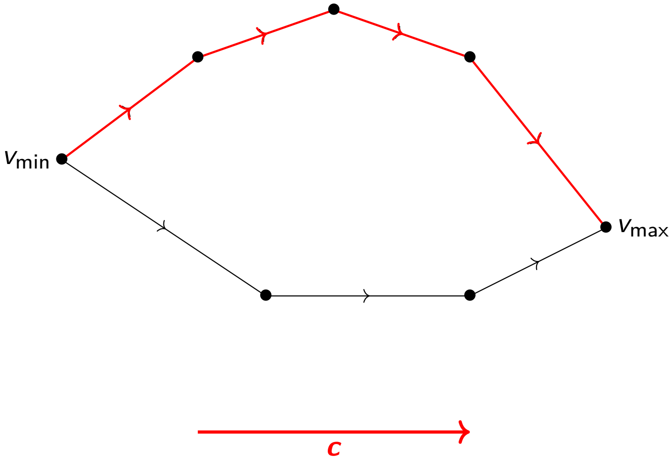
Shadow vertex (pivot) rule

Linear optimization in dimension 2 (simplex method):



Shadow vertex (pivot) rule

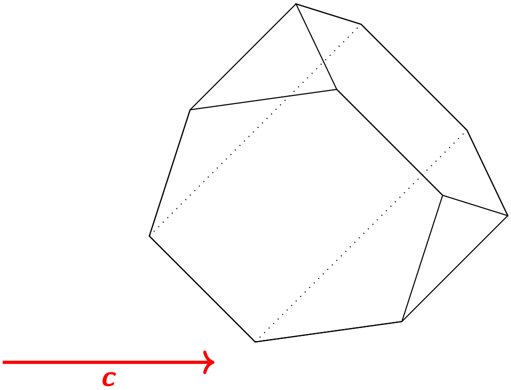
Linear optimization in dimension 2 (simplex method): **EASY** !



Convention: choose upper

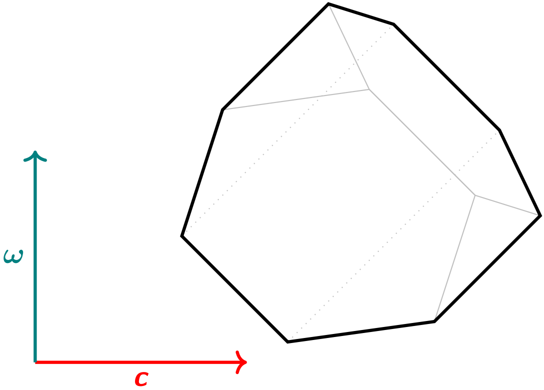
Shadow vertex (pivot) rule

Optimization in higher dimension: make it 2-dimensional!

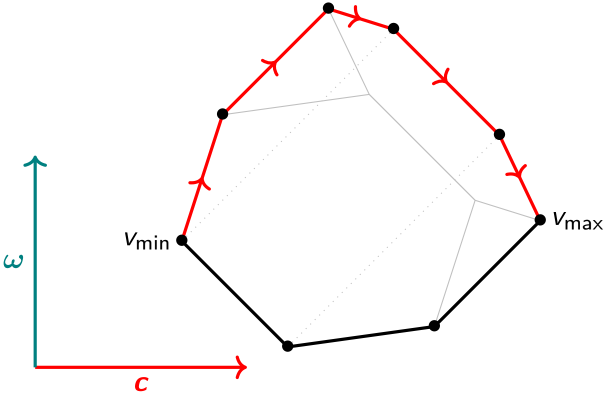


Shadow vertex (pivot) rule

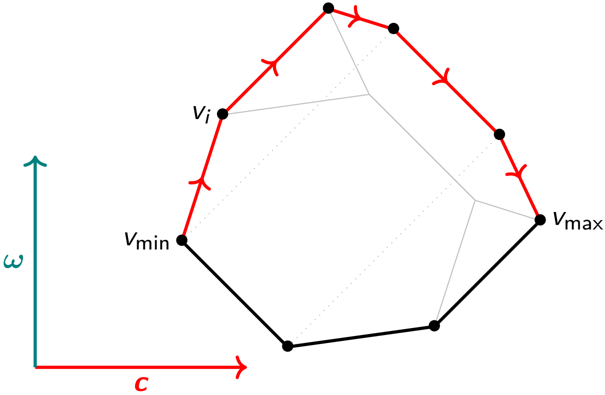
Optimization in higher dimension: make it 2-dimensional!



Optimization in higher dimension: make it 2-dimensional!

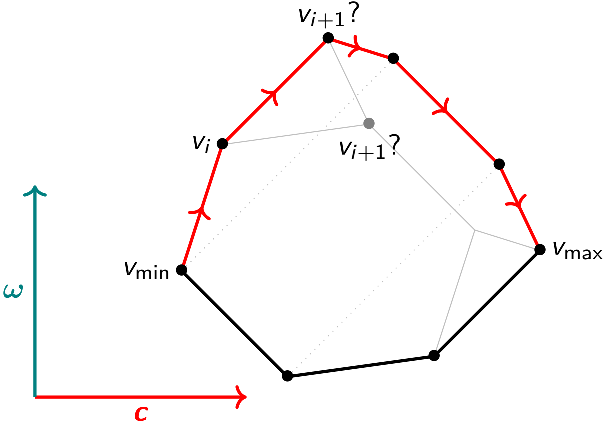


Optimization in higher dimension: make it 2-dimensional!



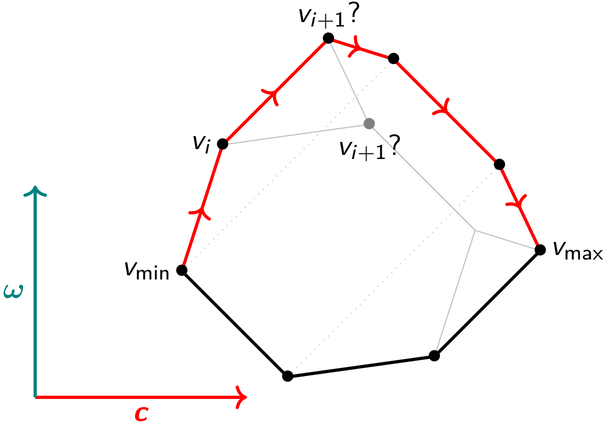
Shadow vertex (pivot) rule

Optimization in higher dimension: make it 2-dimensional!



Shadow vertex (pivot) rule

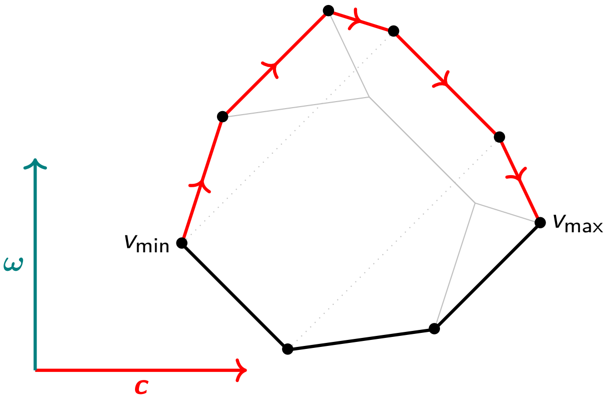
Optimization in higher dimension: make it 2-dimensional!



Shadow vertex rule: take (improving) neighbor with best slope

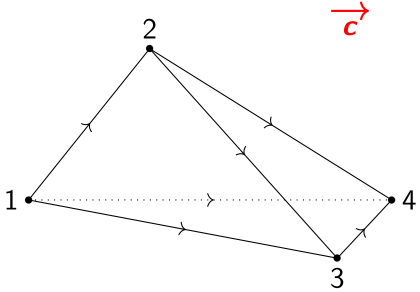
Shadow vertex (pivot) rule

Optimization in higher dimension: make it 2-dimensional!

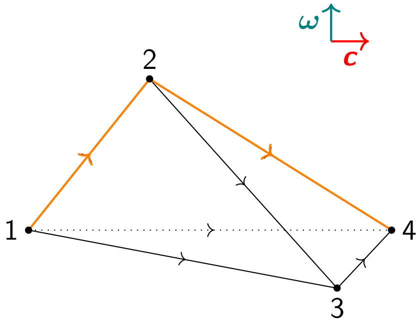


Shadow vertex rule: take (improving) neighbor with best slope
Coherent path: path *captured* by some ω
$N_\ell^{coh} = \#\{\text{coherent paths of length } \ell\}$

Coherent paths of the d-simplex




ω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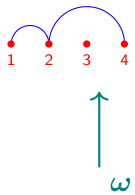
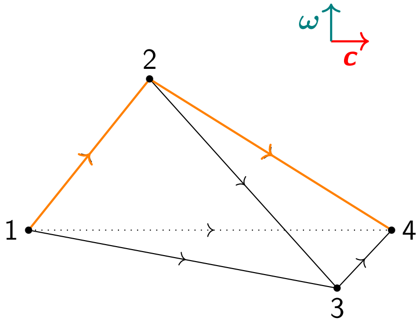
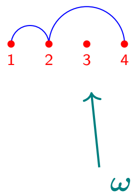
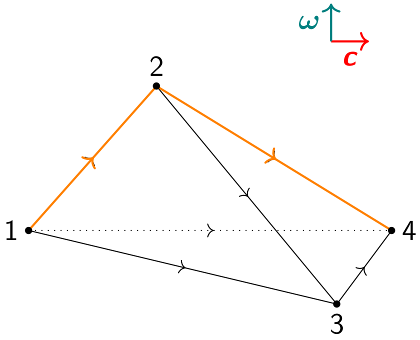
Coherent paths of the d-simplex



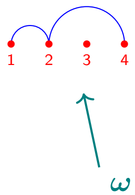


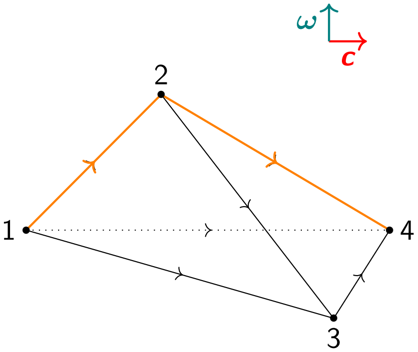
Coherent paths of the d-simplex





Coherent paths of the d-simplex





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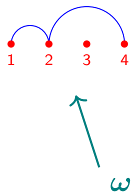

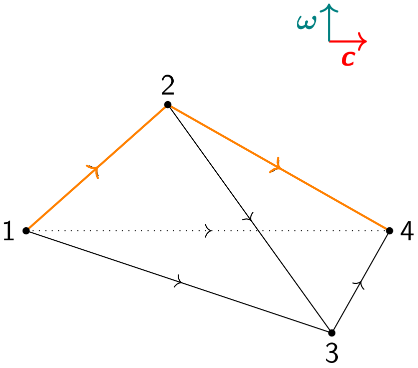

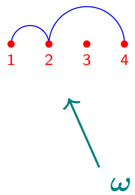
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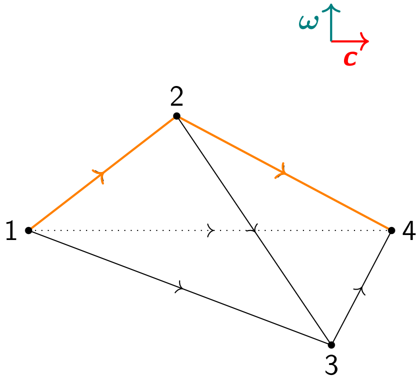
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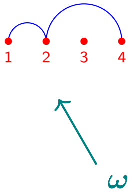
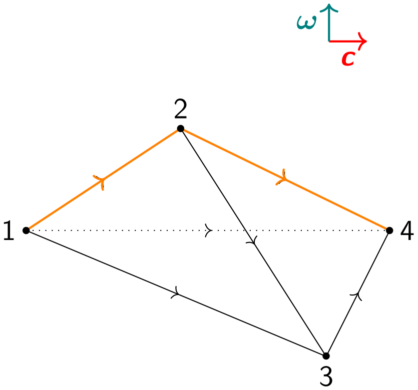
Coherent paths of the d-simplex





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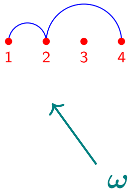

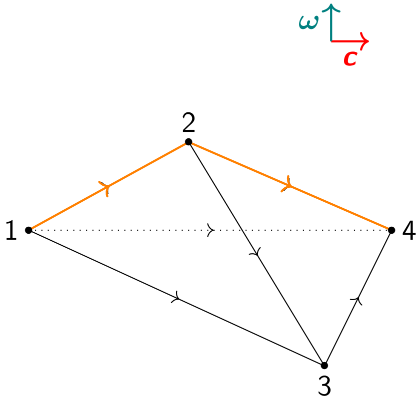

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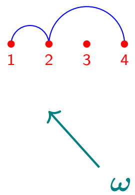

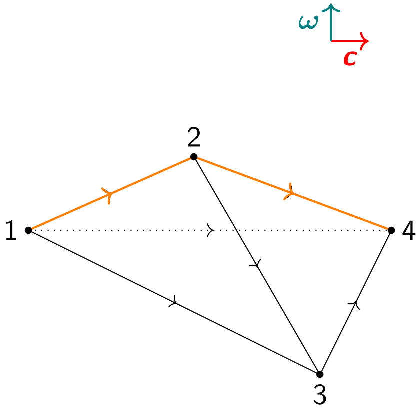

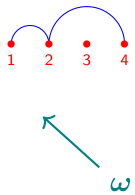
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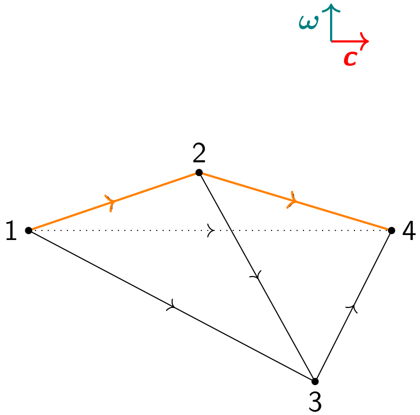
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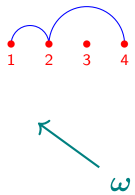

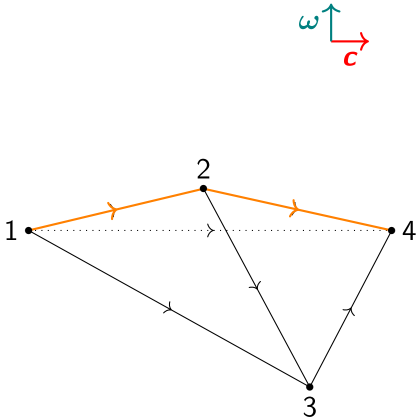

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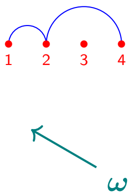

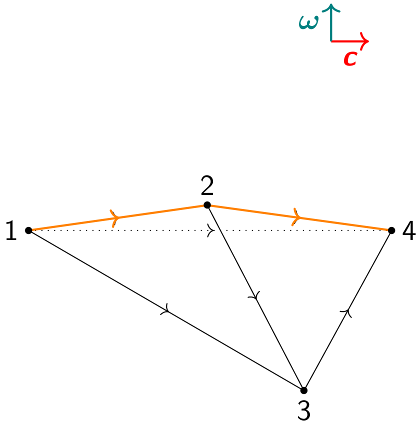

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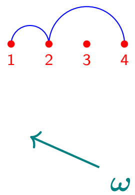

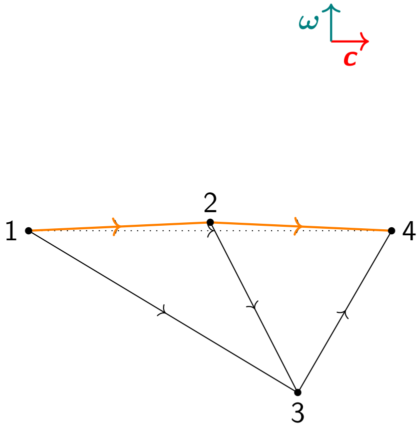

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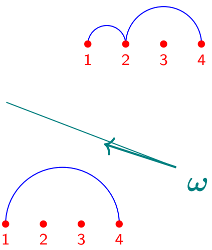

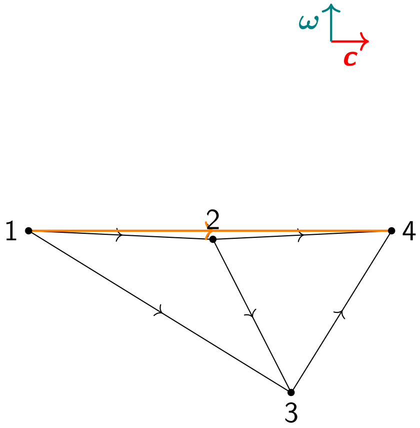

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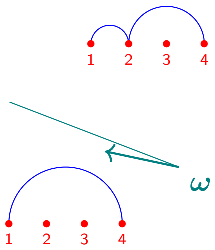

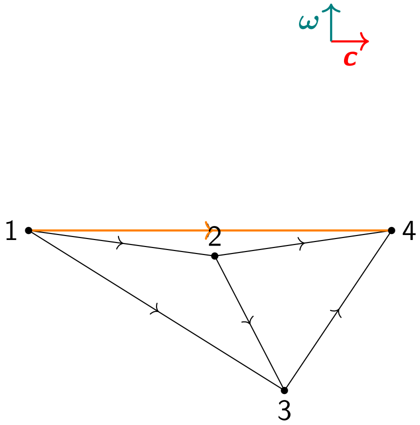

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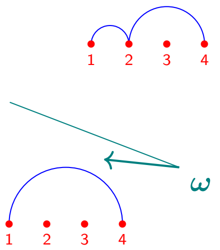

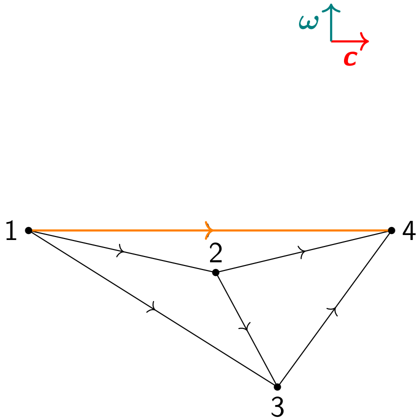

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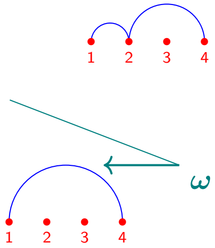
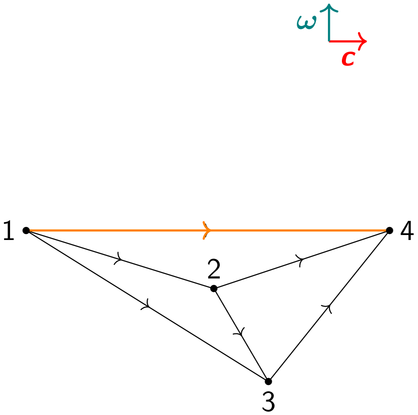
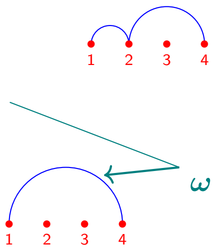
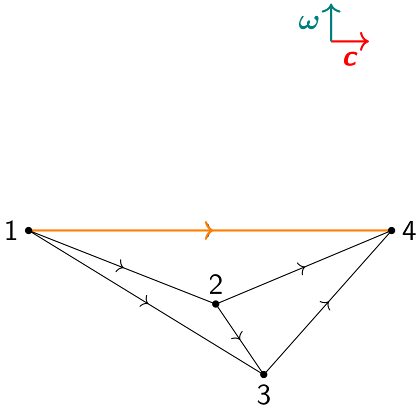
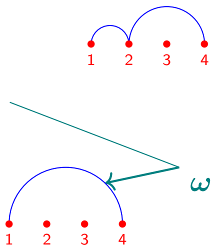




Coherent paths of the d-simplex



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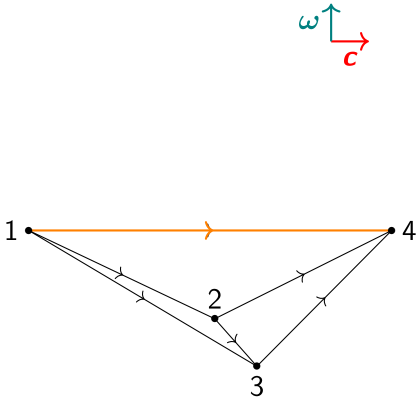
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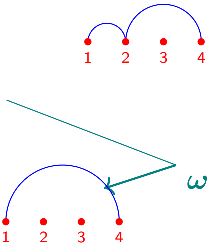
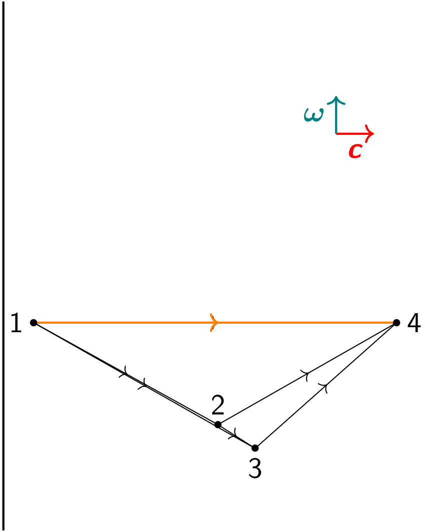
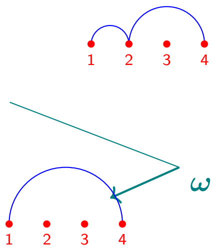
Coherent paths of the d-simplex



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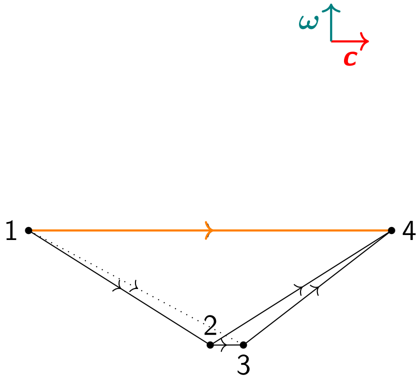
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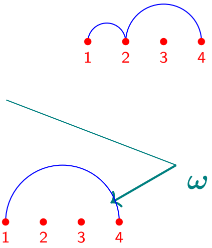
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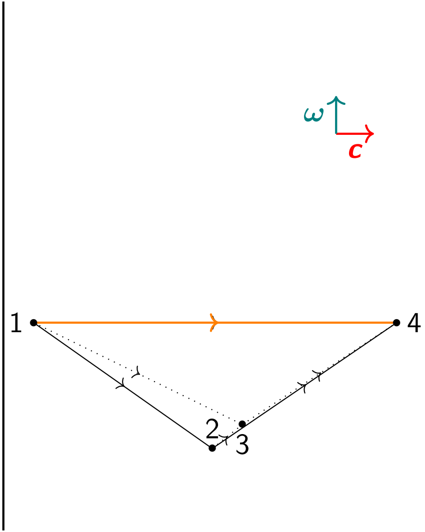
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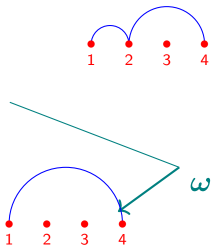
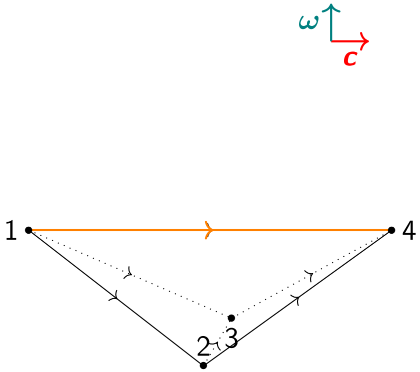
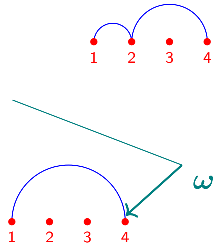
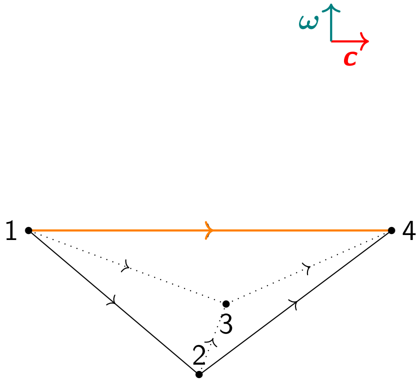
Coherent paths of the d-simplex



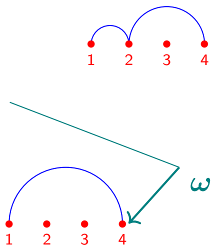


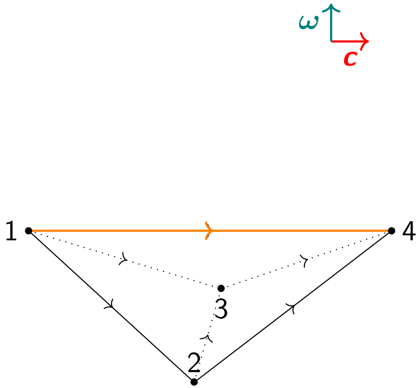
Coherent paths of the d-simplex



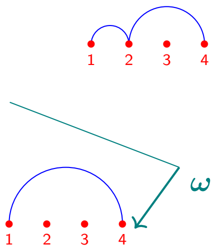


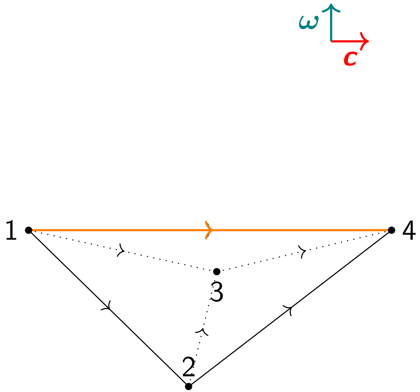
Coherent paths of the d-simplex





Coherent paths of the d-simplex





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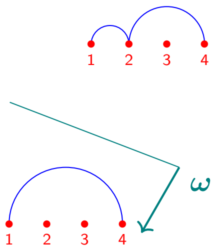

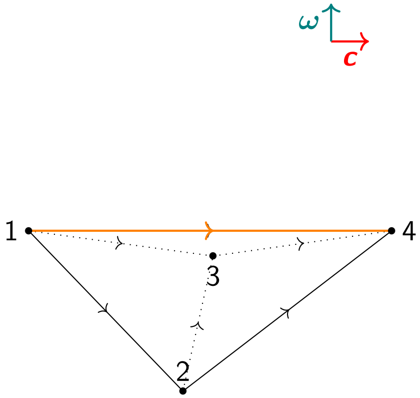

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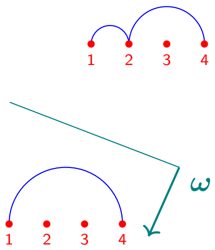

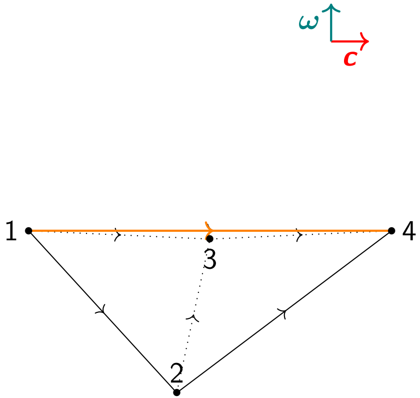

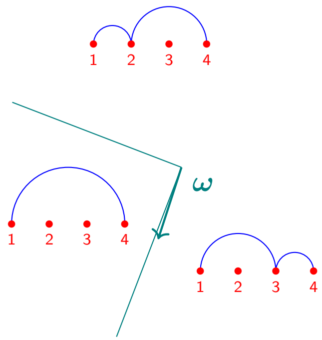
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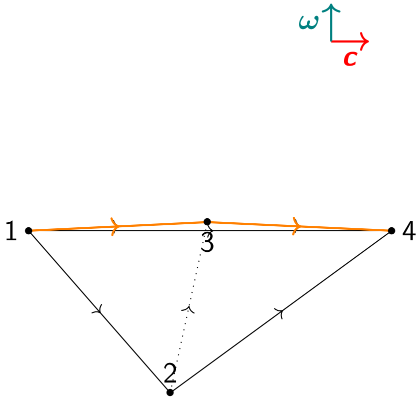
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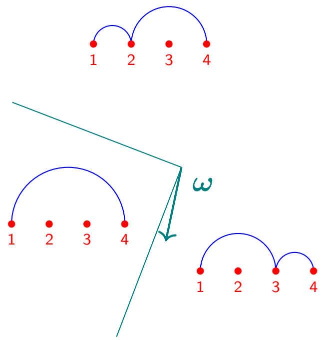
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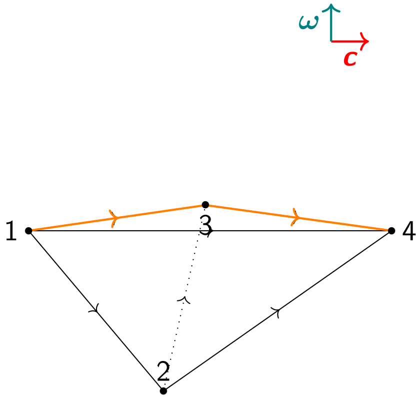
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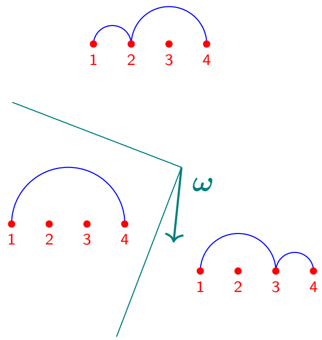
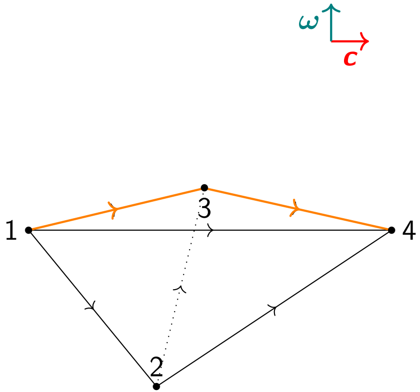
Coherent paths of the d-simplex





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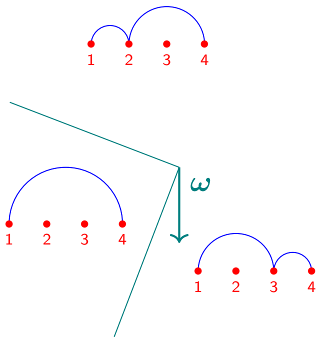

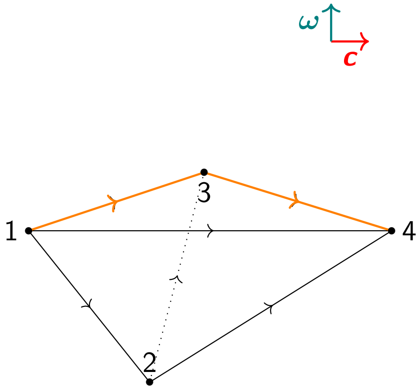

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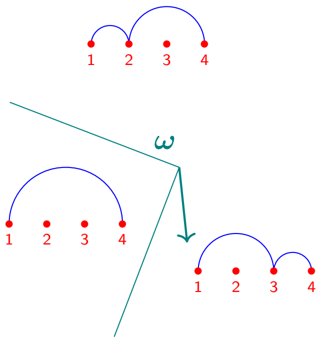

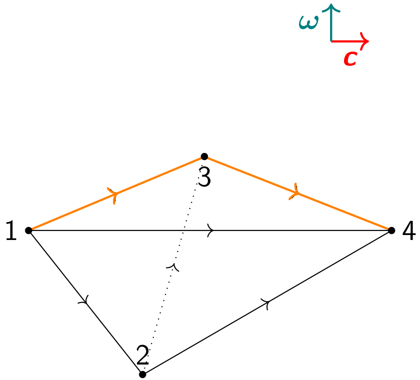

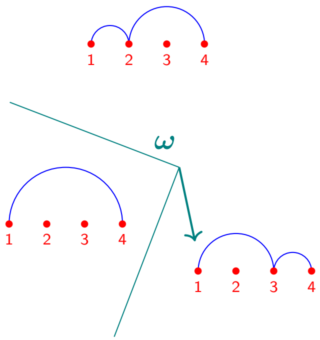
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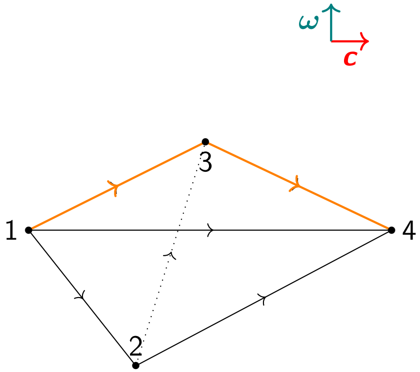
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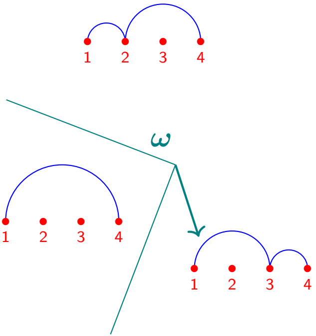
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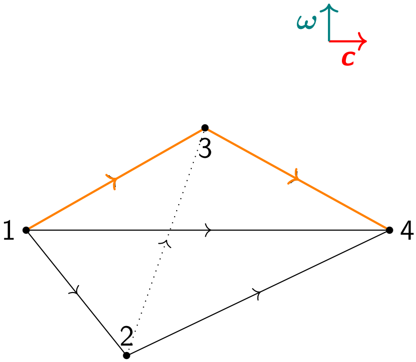
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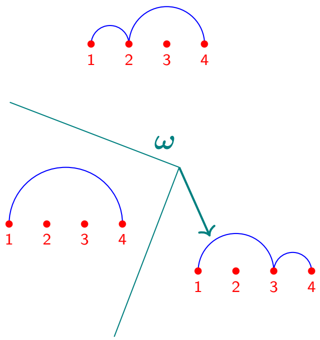
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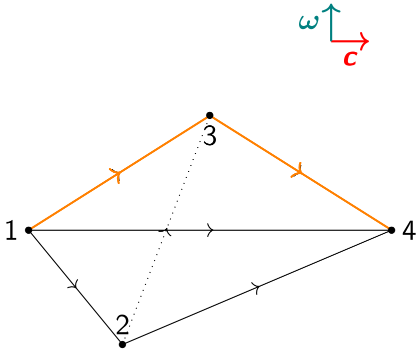
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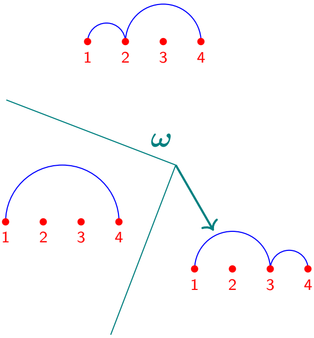
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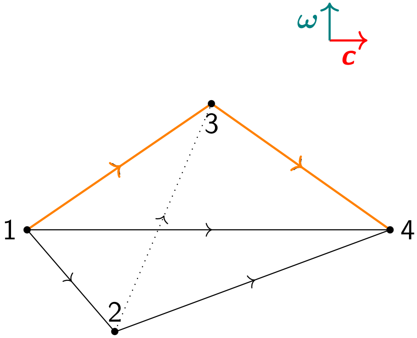
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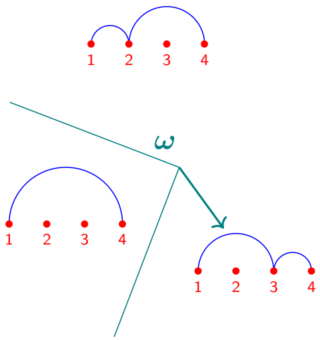
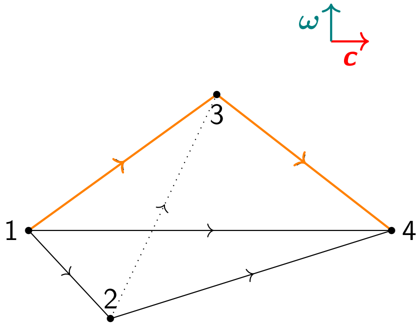
Coherent paths of the d-simplex





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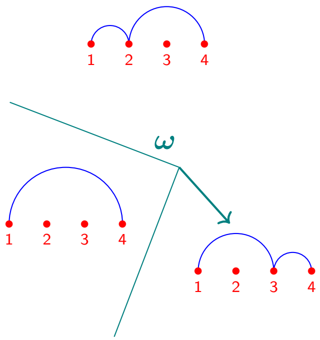

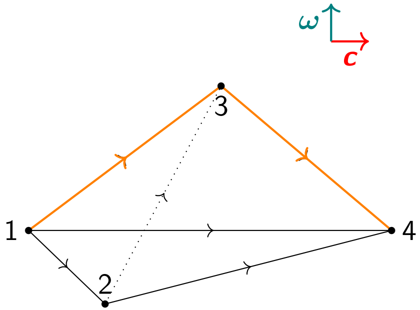

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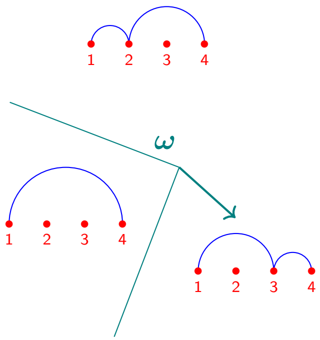

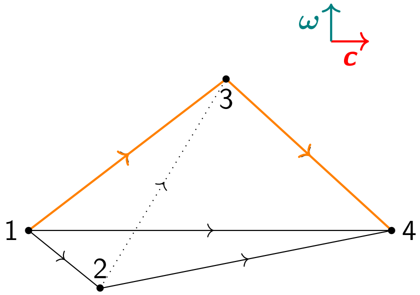

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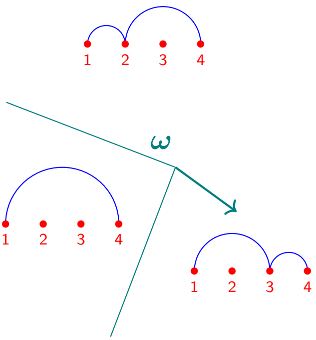

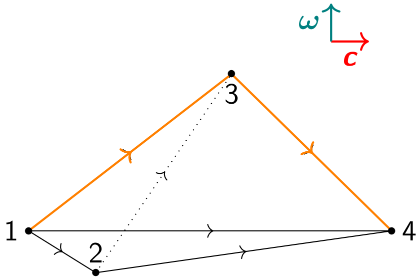

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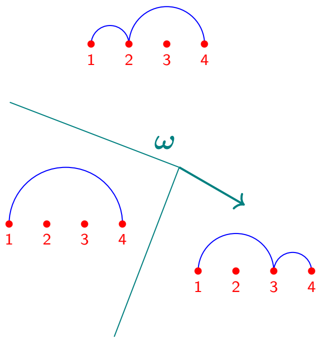

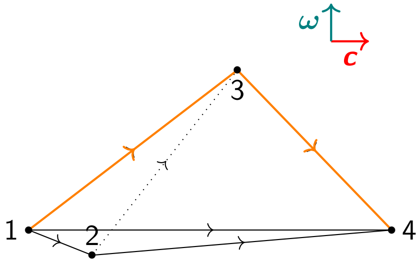

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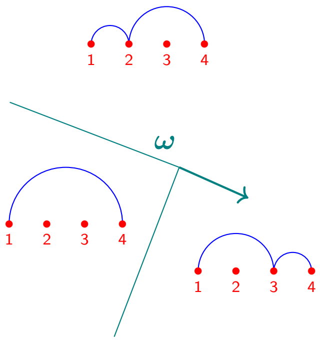

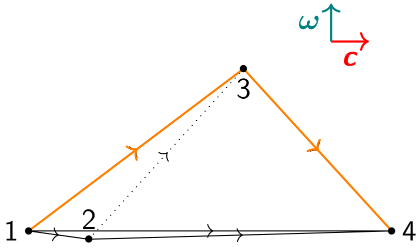

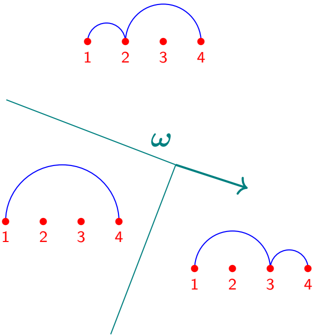
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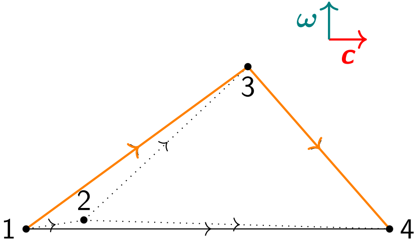
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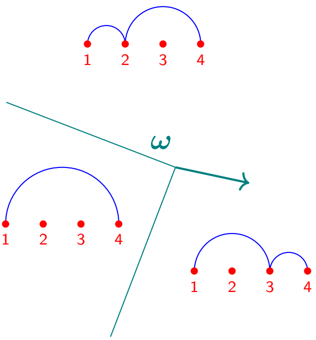
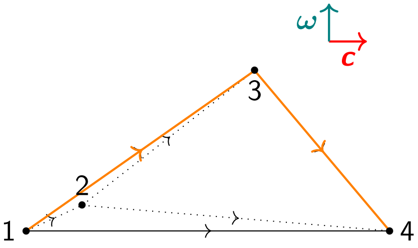
Coherent paths of the d-simplex





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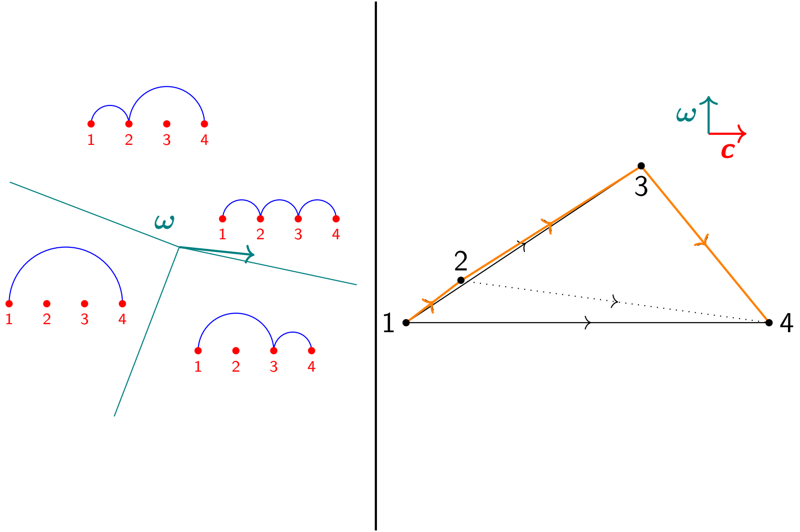

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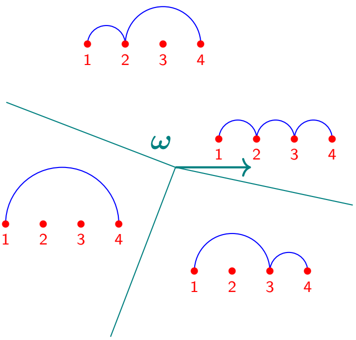

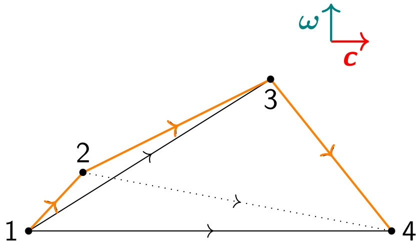

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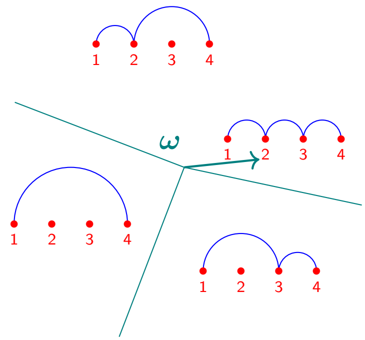

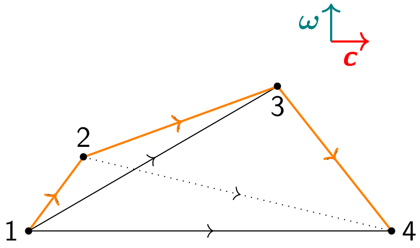

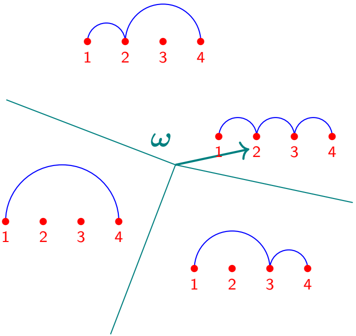
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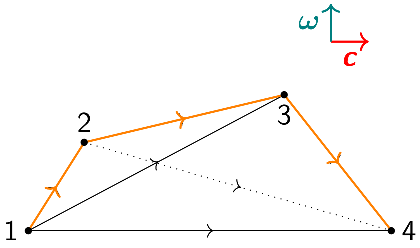
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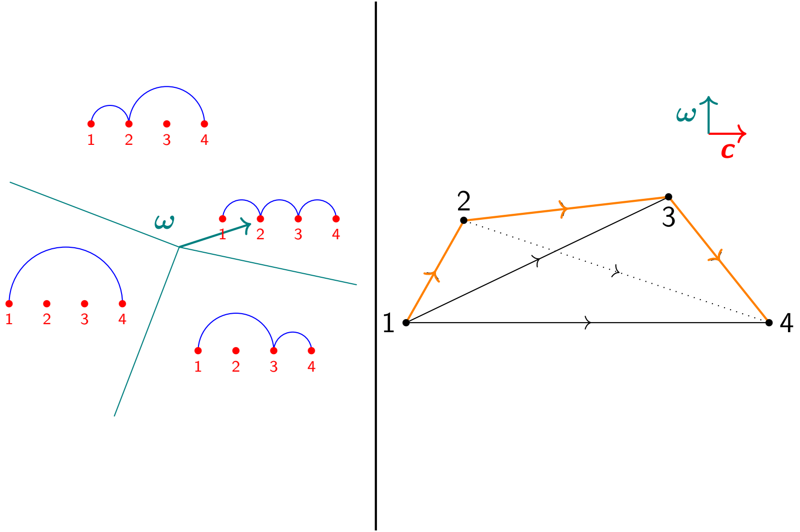

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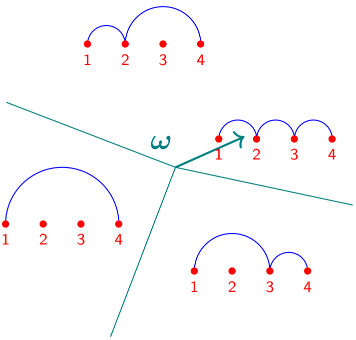

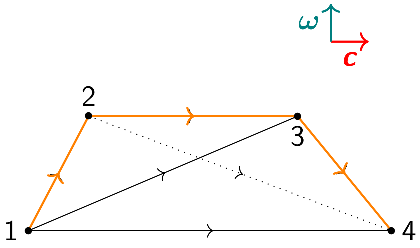

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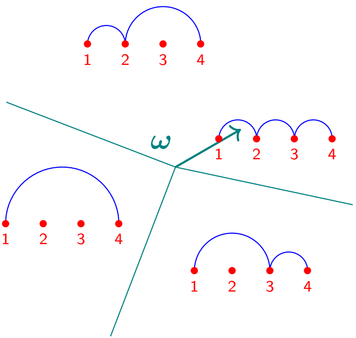

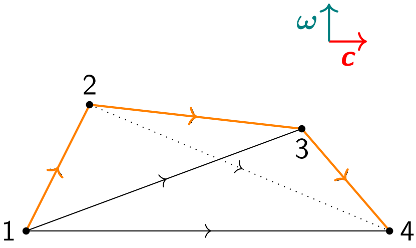

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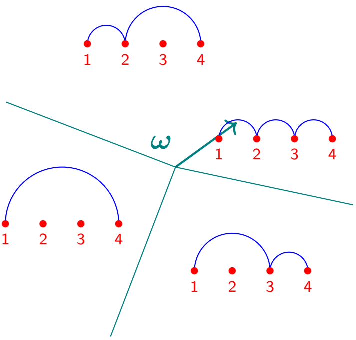

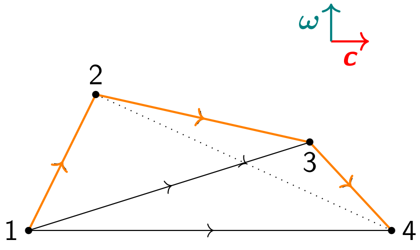

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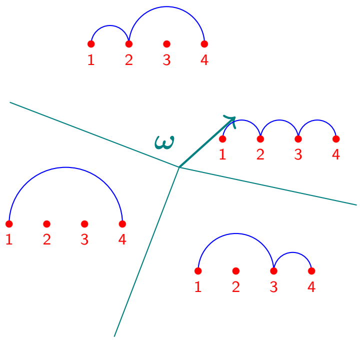

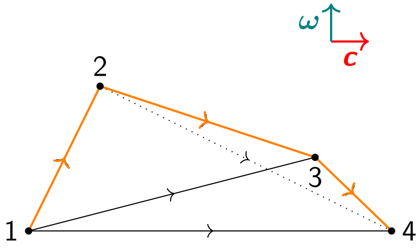

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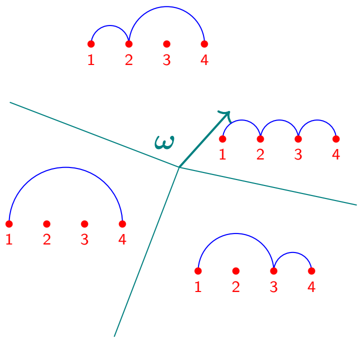

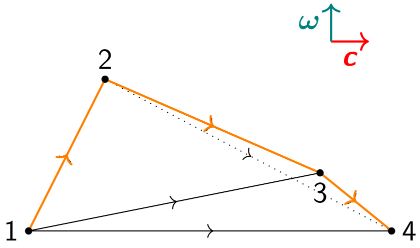

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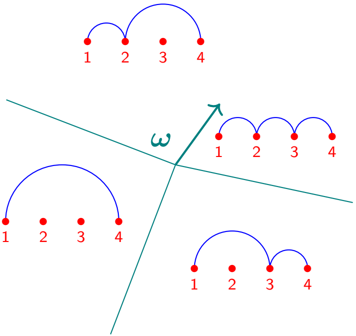

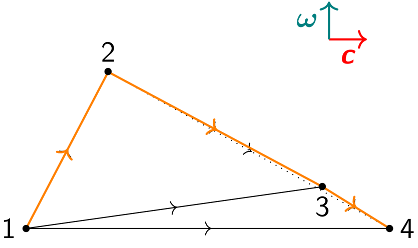

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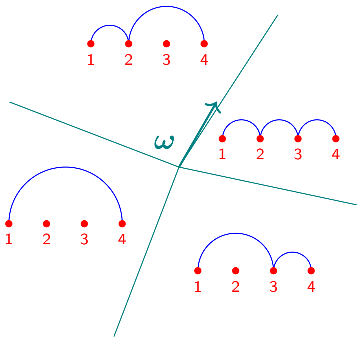

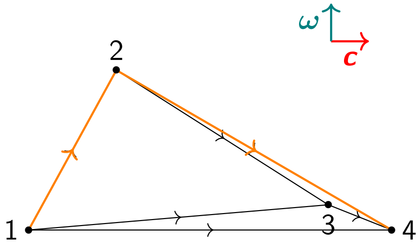

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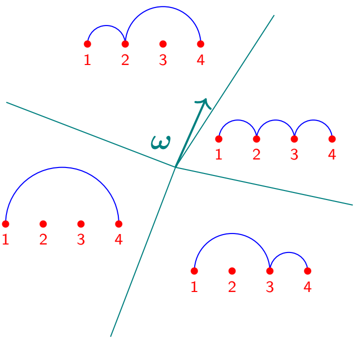

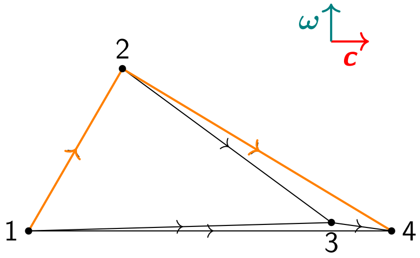

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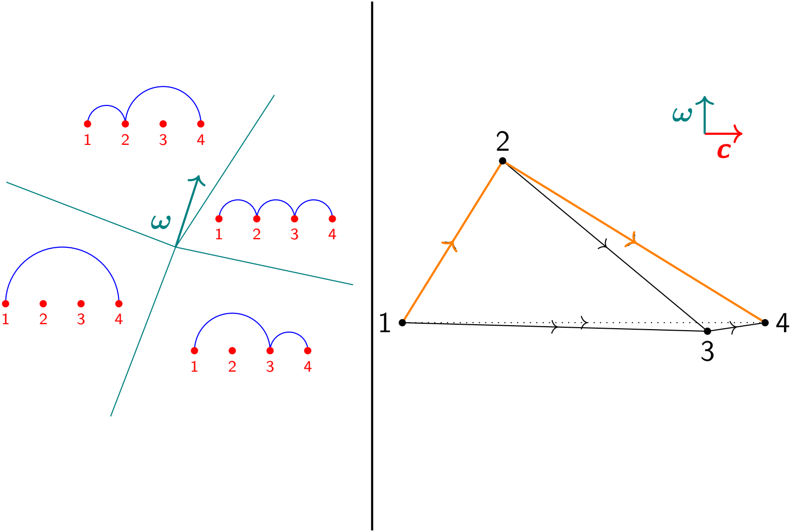

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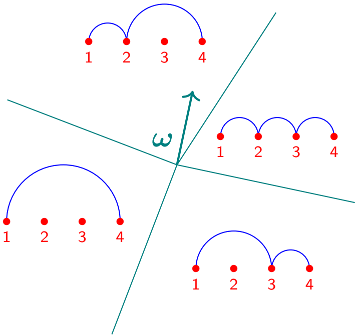

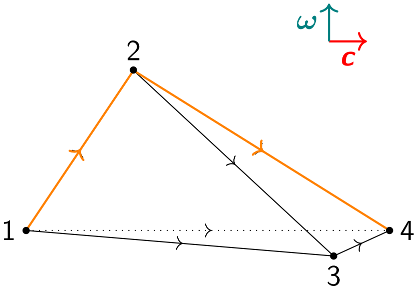

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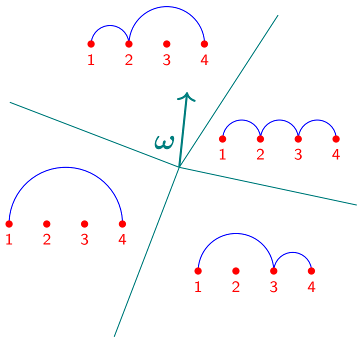
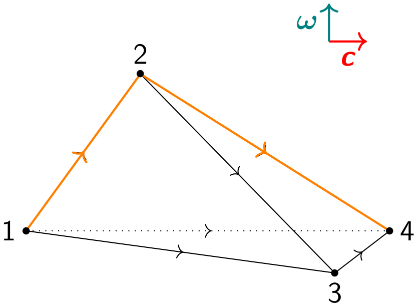
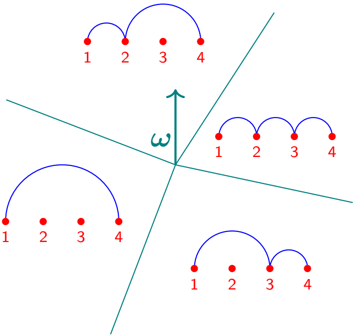
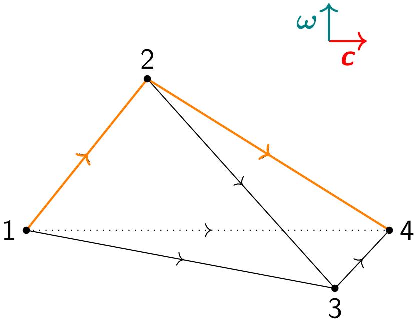




Coherent paths of the d-simplex





Number of paths on a d-simplex

Theorem (Billera–Sturmfels '92 (Fiber polytopes))

For any d-simplex Δ_d and any (generic) $\boldsymbol{c}$

$$N_\ell = N_\ell^{coh} = \binom{d-1}{\ell-1}$$

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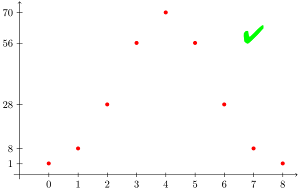

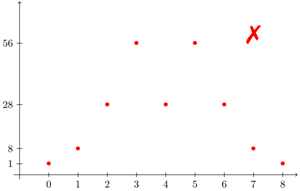

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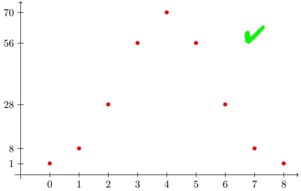
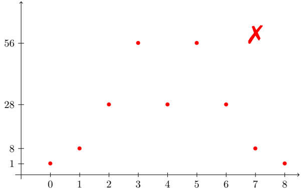
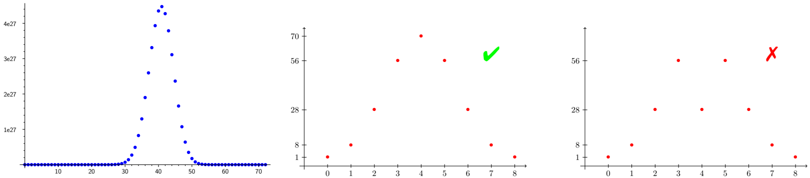




Proof for N_ℓ:
$G_{\Delta_d,\boldsymbol{c}}$: (acyclic) complete graph
Path on Δ_d: sub-set of vertices without $\boldsymbol{v}_{\min}$, $\boldsymbol{v}_{\max}$
Length of path: number of vertices -1

Question A

For P and $\boldsymbol{c}$, are $(N_\ell)_\ell$ and $(N^{\mathrm{coh}}_\ell)_\ell$ unimodal?



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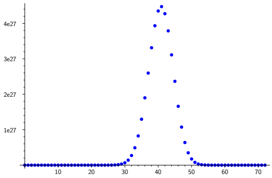
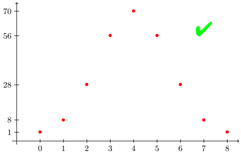
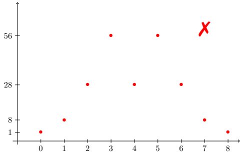




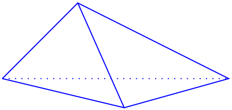


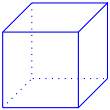
Spoilers:
"Yes" in some cases
"No" in general
"Almost yes" statistically

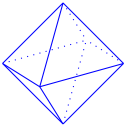
Positive examples

Unimodal examples

polytope	definition
simplex	$\Delta_d := \text{conv}(\boldsymbol{e}_i \ ; \ 1 \leq i \leq d+1)$
cube	$[0,1]^d$
cross-polytope	$\Diamond_d := \text{conv}(\pm \boldsymbol{e}_i \ ; \ 1 \leq i \leq d)$
cyclic polytope	$\text{Cyc}_d(\boldsymbol{t}) := \text{conv}\big((t_i, t_i^2, \dots, t_i^d) \ ; \ 1 \leq i \leq n\big)$
S-hypersimplex	$\Delta_d(S) := \text{conv}(\boldsymbol{x} \in \{0,1\}^d \ ; \ \sum_i x_i \in S)$



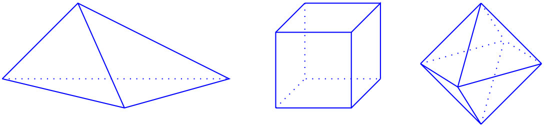




...

Unimodal examples

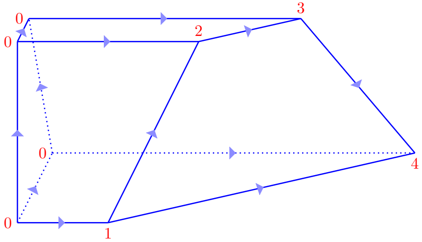
polytope	N_ℓ	N_ℓ^{coh}
simplex	$\binom{d-1}{\ell-1}$	
cube	$d!$ iff $\ell = d$	
cross-polytope	$2\sum_{k=0}^{d-2}\binom{2k}{\ell-2}$	$\binom{d-1}{\ell-1}2^{\ell-1}$
cyclic polytope	$\binom{n-2}{\ell-1}$	complicated...
S-hypersimplex	$\binom{d}{\tilde{s}_1,\tilde{s}_2,\dots,\tilde{s}_r}$ iff $\ell = \lvert S\rvert$	

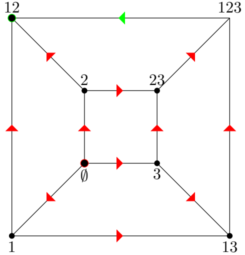


. . .

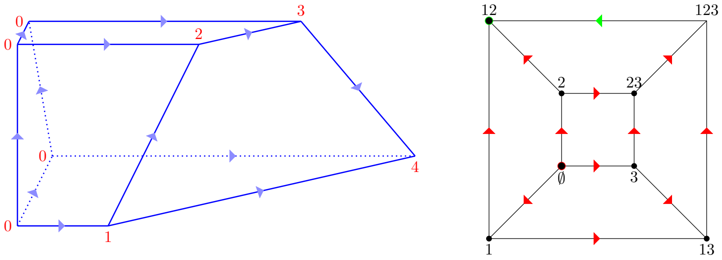
Sources: Billera–Sturmfels '92, Athanasiadis–De Loera–Reiner '00, Maneck–Sanyal–So '20, Black–De Loera '23 + our computations

Negative examples


0
0
3
2
0
4
0
1

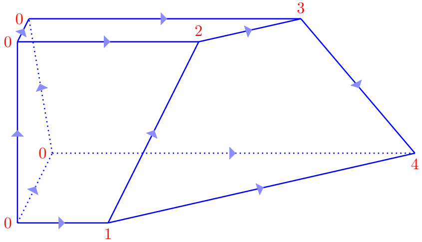

12
123
2
23
∅
3
1
13

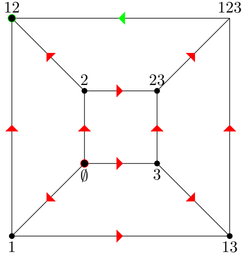
Lopsided d-cube



ℓ	2	3	4
$N_\ell = N_\ell^{\text{coh}}$	2	0	4

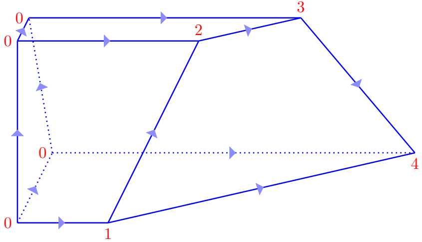
Lopsided d-cube

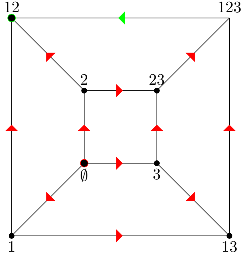




ℓ	$d-1$	d	$d+1$
$N_\ell = N_\ell^{\text{coh}}$	$(d-1)!$	0	$\frac{1}{6}(d+1)!$

Works in any dimension.





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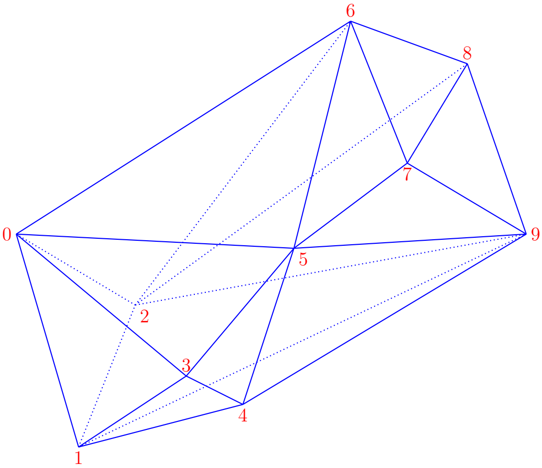
Works in any dimension.

N.B.: One can remove 0s.

Simplicial counter-example



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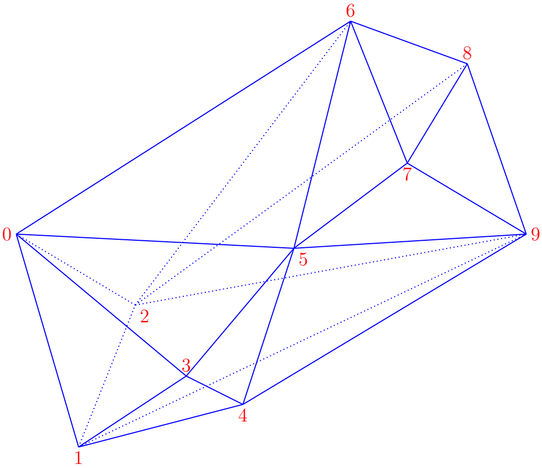

ℓ	2	3	4	5	6	7	8	total
N_ℓ	3	8	12	11	12	6	1	53

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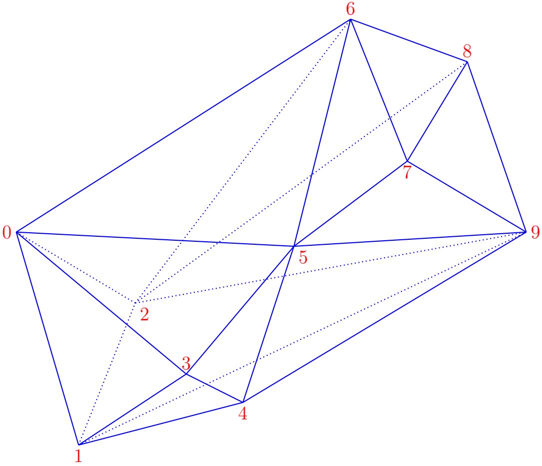

ℓ	2	3	4	5	6	7	8	total
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N.B.: One can put the vertices on a sphere

Loday's associahedron of dimension 5

Definition

Generalized permutahedron: all edges of P are in direction $\boldsymbol{e}_i - \boldsymbol{e}_j$ for some $i \neq j$

Loday's associahedron, '04

Asso_n is a generalized permutahedron

$$\mathrm{Asso}_n = \left\{ \boldsymbol{x} \in \mathbb{R}^n \; ; \; \begin{array}{l} \sum_{i=1}^{n} x_i = 0 \\ \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \quad \text{for} \quad \emptyset \neq I = [a, b] \subsetneq [n] \end{array} \right\}$$

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For Asso_5: $\boldsymbol{c} = (1, 2, 3, 4, 5)$

ℓ	6	7	8	9	10	11	12	13	14	15	16
N_ℓ	1	20	112	232	382	348	456	390	420	334	286
N_ℓ^{coh}	1	20	105	206	332	274	332	270	206	122	142

Sources: Nelson '17 + our computations

Polytopes with 0/1-coordinates

Definition (0/1-polytopes)

For $\mathcal{X} \subseteq 2^{[n]}$, define $\mathrm{P}_{\mathcal{X}} := \mathrm{conv}\,(\boldsymbol{e}_X \;;\; X \in \mathcal{X})$
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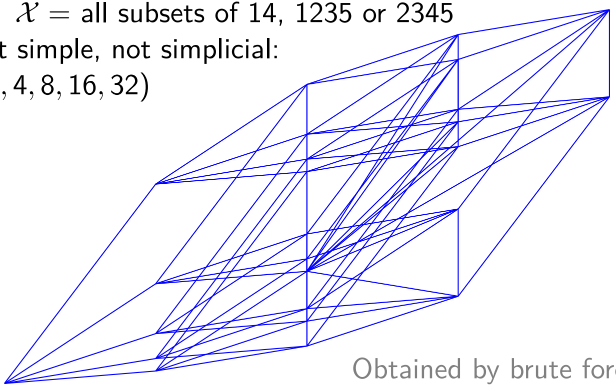
$n = 5$, $\mathcal{X}$ = all subsets of 14, 1235 or 2345
$\mathrm{P}_{\mathcal{X}}$ not simple, not simplicial:
$\boldsymbol{c} = (2, 4, 8, 16, 32)$



Obtained by brute force

ℓ	3	4	5	6	7	8	total
N_ℓ	2	36	96	76	84	36	330

Random case

A bit of (hi)story

Pick a model of random polytopes   *e.g.*, n points on $\mathbb{S}^{d-1}$

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Gave: Formulae $\mathbb{E}L_n$ for several models
Asked: **What are the higher moments?** *e.g.*, variance

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$\rightsquigarrow$ I want a central limit theorem:
$n \to +\infty \Rightarrow$ (almost) $L_n \sim \mathcal{N}(0, 1)$ once normalized
dimension d is fixed

I won't spreak on unimodality in random model: see details under the rug!

Uniform measure on the sphere

$X_1, \ldots, X_n \sim \mathcal{U}(\mathbb{S}^{d-1})$ $\qquad$ $\mathrm{P}_n = \mathrm{conv}(X_1, \ldots, X_n)$

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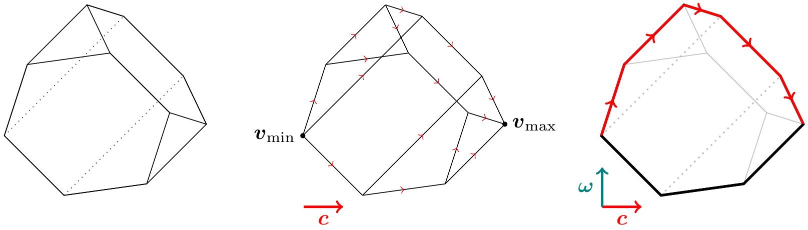

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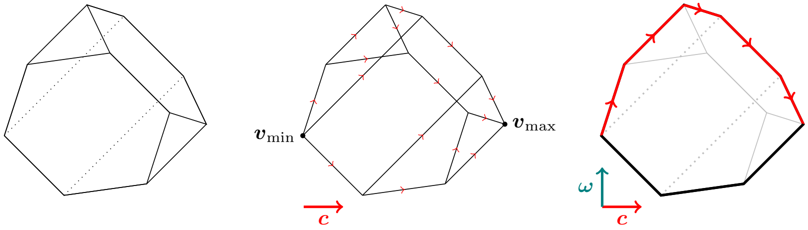
Z_i = 2-dim projection of X_i $\quad\quad$ $\mathrm{Q}_n = \mathrm{conv}(Z_1, \ldots, Z_n)$

I want: $f_0(\mathrm{Q}_n)$ or $f_1(\mathrm{Q}_n)$



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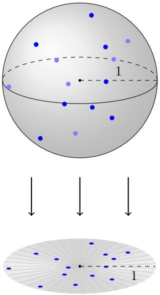

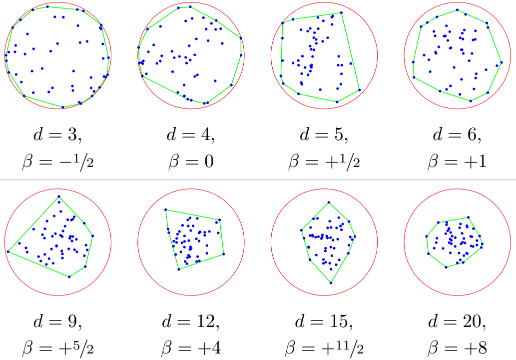

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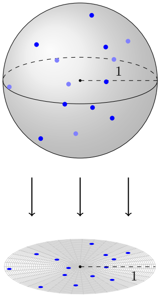

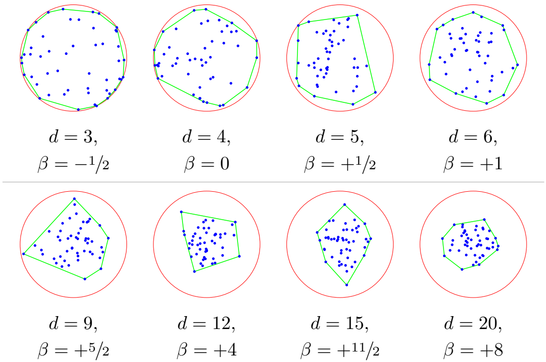

$d \nearrow \;\Rightarrow$ concentration around center

β-polygons

$X_1, \dots, X_n \sim \mathcal{U}(\mathbb{S}^{d-1})$ $\quad Z_i =$ 2-dim proj X_i $\quad \mathrm{Q}_n = \mathrm{conv}(Z_i)_i$

Theorem (Kabluchko–Thäle–Zaporozhets '20)

Z_i distributed according to density:

$$f_{2,\beta_d}(\boldsymbol{x}) = C\left(1 - \|\boldsymbol{x}\|^2\right)^{\beta_d} \qquad \text{for } \boldsymbol{x} \in \mathbb{B}^2$$

where $\beta_d = \frac{1}{2}d - 2$ and C is a constant

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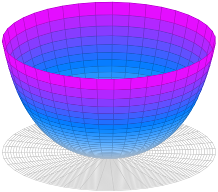
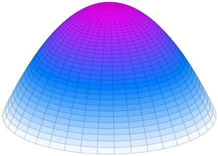
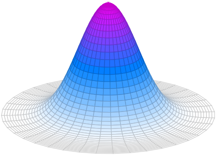


$\beta = -0.5$



$\beta = +1$



$\beta = +10$

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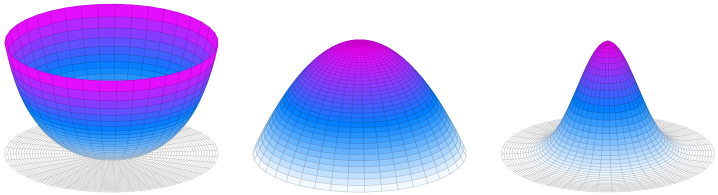


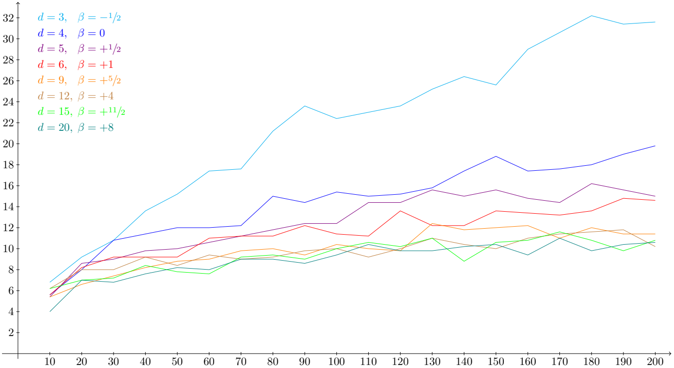
$\beta = -0.5$ $\quad\quad$ $\beta = +1$ $\quad\quad$ $\beta = +10$

N.B.: $d = 3 \Leftrightarrow \beta < 0$; $d = 4 \Leftrightarrow \mathcal{U}(\mathbb{B}^2)$; $d \geq 5 \Leftrightarrow \beta > 0$

In the following: $d \geq 4$

Expectancy

$Z_1, \dots, Z_n \sim \beta$-distributed $\quad \beta_d = \frac{1}{2}d - 2 \quad Q_n = \mathrm{conv}(Z_i)_i$



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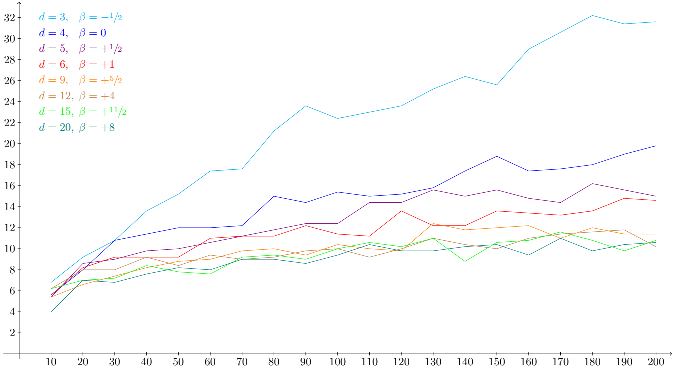

Theorem (Kabluchko–Thäle–Zaporozhets '20)

$$\mathbb{E} f_0(\mathrm{Q}_n) \sim c\, n^{\frac{1}{d-1}}$$

where $c > 0$ is a constant (some-what explicit)

Variance

$Z_1, \ldots, Z_n \sim \beta$-distributed $\quad \mathrm{Q}_n = \mathrm{conv}(Z_i)_i \quad \mathbb{E} f_0(\mathrm{Q}_n) \sim c\, n^{\frac{1}{d-1}}$

Theorem (Juhnke–P. '25)

$$c'\, n^{\frac{1}{d-1} - a} \leq \mathrm{Var}\, f_0(\mathrm{Q}_n) \leq c''\, n^{\frac{1}{d-1}}$$

for all $a > 0$*, for some constants* c', c''

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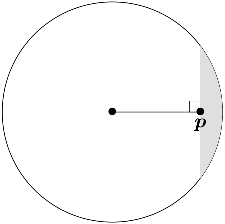
Proofs' ideas

Lower bound: ε-floating body + kind of Sylvester's 4-point problem

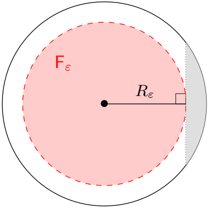
Upper bound: 1^{st} order difference + Efron–Stein jackknife ineq.

ε-cap: cap with measure $= \varepsilon$ Careful: measure accoring to β-density

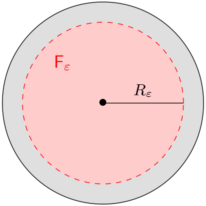
ε-floating body: complement of all ε-caps



C_p: gray region



R_ε and F_ε satisfy:
measure of the gray region $= \varepsilon$



$\mathbb{P}(X \notin F_\varepsilon) = \mu(\mathbb{B}^2 \setminus F_\varepsilon)$
$=$ measure of the gray region

ε-cap and ε-floating body

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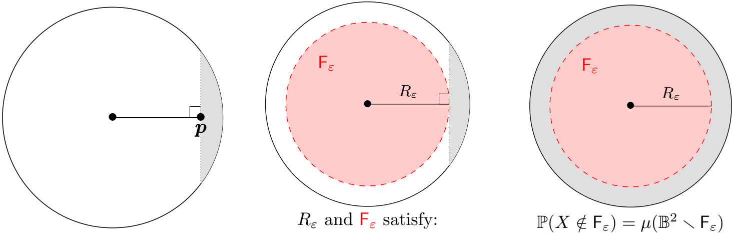

$C_{\boldsymbol{p}}$: gray region

R_ε and F_ε satisfy: measure of the gray region $= \varepsilon$

$\mathbb{P}(X \notin \mathsf{F}_\varepsilon) = \mu(\mathbb{B}^2 \smallsetminus \mathsf{F}_\varepsilon)$ $=$ measure of the gray region

Lemma

For any $s > 0$ and $\varepsilon = c_0 \frac{\log n}{n}$, with $c_0 = \frac{1}{d-1} + s$

$$\mathbb{P}(\mathsf{F}_\varepsilon \subseteq \mathsf{Q}_n) \geq 1 - n^{-s}$$

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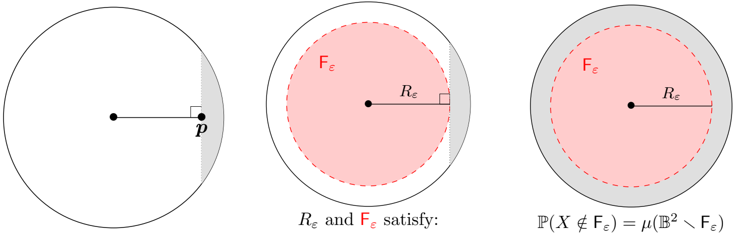

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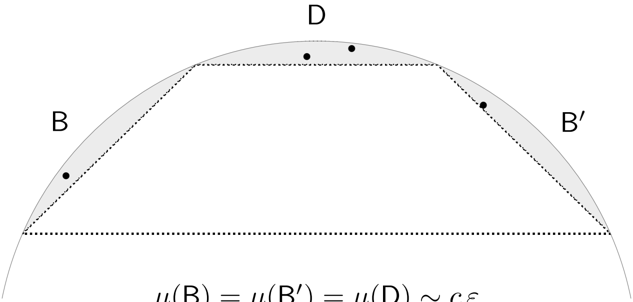
Lemma

If $\varepsilon = c_0 \frac{\log n}{n}$, there "is" ≥ 1 vertex of Q_n in each ε-cap

Sylvester 4-point problem



$$\mu(\mathsf{B}) = \mu(\mathsf{B}') = \mu(\mathsf{D}) \sim c\,\varepsilon$$

$\mathbb{P}\Big(\text{conv(these 4 points) is triangle}\Big)$ far from 0 and 1 indepently of ε

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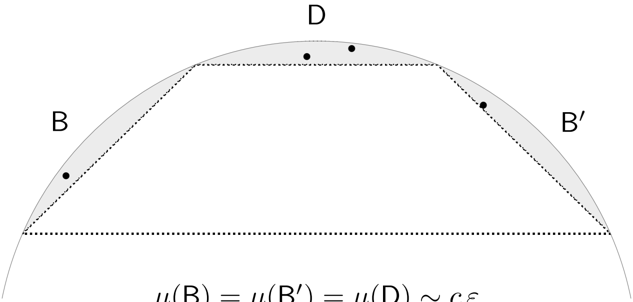

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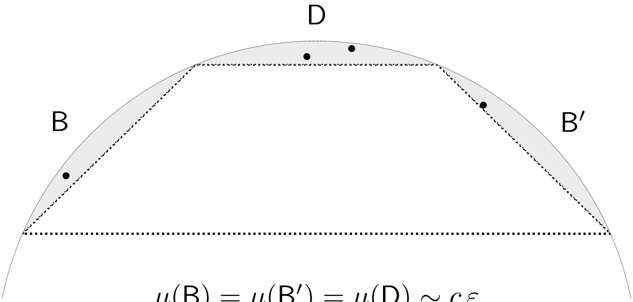

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Corollary (Juhnke–P. '25)

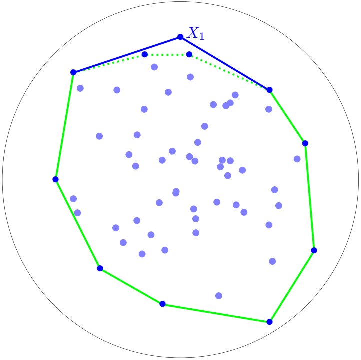
$$\mathrm{Var}\, f_0(\mathrm{Q}_n) \;\geq\; \mathbb{E}(\mathrm{Var}(f_0(\mathrm{Q}_n) \mid \boldsymbol{X})) \;\geq\; c'\, n^{\frac{1}{d-1} - c_0}$$

First order difference operator

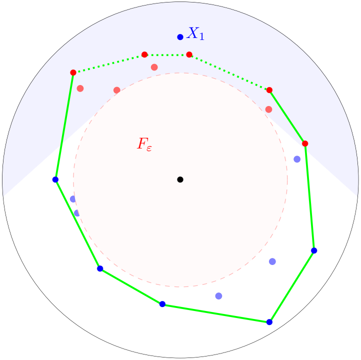
Definition

First order difference operator for $f = f_0(\text{conv}(\dots))$

$$Df(X_1, X_2, \dots, X_n) = f(X_1, X_2, \dots, X_n) - f(X_2, \dots, X_n)$$

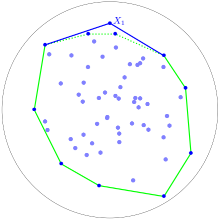


The dotted edges are edges of Q_n, but not of Q_{n+1}



Red points are visible from X_1: we control pairs of them.

Efron–Stein jackknife inequality



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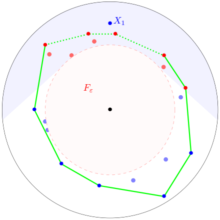

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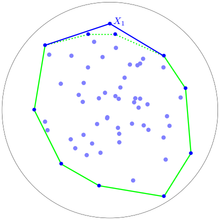
Theorem (Efron–Stein jackknife inequality)

$$\operatorname{Var} f_0(Q_n) \leq (n+1)\,\mathbb{E}\Big(\big(Df_0(Q_{n+1})\big)^2\Big)$$



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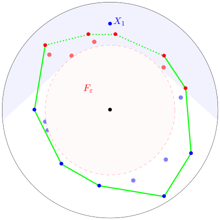

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Theorem (Efron–Stein jackknife inequality)

$$\operatorname{Var} f_0(\mathsf{Q}_n) \leq (n+1)\,\mathbb{E}\Big(\big(Df_0(\mathsf{Q}_{n+1})\big)^2\Big)$$

Theorem (Juhnke–P. '25)

For $p \geq 1$ integer, there is $c > 0$:

$$\mathbb{E}\big(|Df_0(\mathsf{Q}_n)|^p\big) \leq c\,(\log n)^{p+1-\frac{1}{d-1}} \left(\frac{1}{n}\right)^{1-\frac{1}{d-1}} \qquad \textit{if } n \to +\infty$$

Corollary: $\operatorname{Var} f_0(\mathsf{Q}_n) \leq c''\, n^{\frac{1}{d-1}}$ for some $c'' > 0$

Central limit theorem using Kolmogorov distance

Kolmogorov distance $d_{\mathrm{Kol}}(X,\ Y) = \sup_{x\in\mathbb{R}} \left|\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)\right|$

Theorem (Central limit theorem, Juhnke–P. '25)

With $U \sim \mathcal{N}(0,1)$

$$d_{Kol}\left(\frac{f_0(\mathrm{Q}_n) - \mathbb{E}f_0(\mathrm{Q}_n)}{\sqrt{\mathrm{Var}\, f_0(\mathrm{Q}_n)}},\ U\right) \quad \leq \quad c\,(\log n)^{\frac{7}{2}-\frac{1}{2(d-1)}}\ \left(\frac{1}{n}\right)^{\frac{1}{2(d-1)}}$$

$$\to \quad 0 \qquad \textit{when } n \to +\infty$$

Controlling the Kolmogorov distance

Definition

Second order difference operator for $f = f_0(\mathrm{conv}(\dots))$

$$D_{12}f(X_1, X_2, X_3, \dots, X_n) = f(X_1, X_2, X_3 \dots, X_n) \\ -f(X_2, X_3, \dots, X_n) - f(X_1, X_3, \dots, X_n) \\ +f(X_3, \dots, X_n)$$

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Theorem (Shao–Zhang's '25 & Lachièze-Rey–Peccati '17)

$$d_{\mathrm{Kol}}\left(\frac{W - \mathbb{E}(W)}{\sqrt{\mathrm{Var}\, W}}, U\right) \leq c \frac{1}{\mathrm{Var}\, W}\left(\sqrt{n}\gamma_1 + n\sqrt{\gamma_2} + n\sqrt{n}\sqrt{\gamma_3}\right)$$

Controlling the Kolmogorov distance

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$$\gamma_1(f) = \mathbb{E}\Big(|Df(\boldsymbol{X})|^4\Big)$$

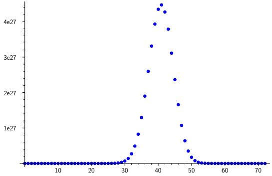
Do not read!

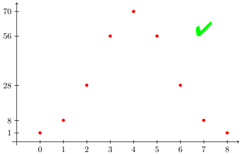
$$\gamma_2(f) = \sup_{(\boldsymbol{Y},\boldsymbol{Z})} \mathbb{E}\Big(\mathbf{1}(D_{12}f(\boldsymbol{Y}) \neq 0)\, D_1 f(\boldsymbol{Z})^4\Big)$$

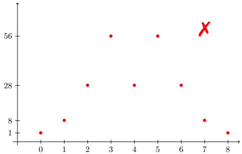
$$\gamma_3(f) = \sup_{(\boldsymbol{Y},\boldsymbol{Y}',\boldsymbol{Z})} \mathbb{E}\Big(\mathbf{1}(D_{12}f(\boldsymbol{Y}) \neq 0)\, \mathbf{1}(D_{13}f(\boldsymbol{Y}') \neq 0)\, D_2 f(\boldsymbol{Z})^4\Big)$$

Question A

For P and $\boldsymbol{c}$, are $(N_\ell)_\ell$ and $(N_\ell^{\text{coh}})_\ell$ always unimodal?







~~Spoilers:~~ Answers:

Conclusion

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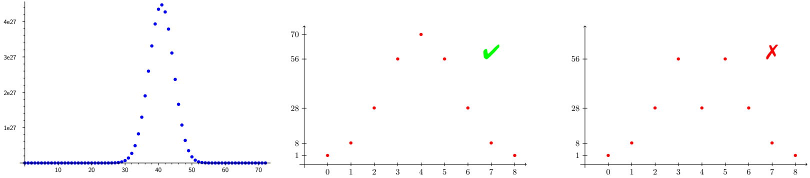

~~Spoilers:~~ Answers:

"Yes" in meaningful but highly symmetric cases

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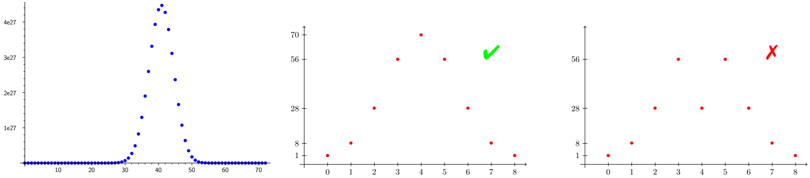

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"No" in all dim, for simple, simplicial, edge-restriction, 0/1...

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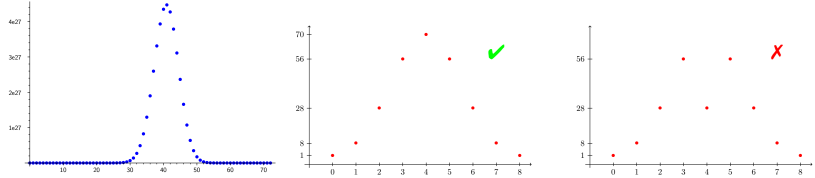

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Length admits central limit theorem, i.e. histogram near Gaussian

for 1 natural model

Thank you!